Pergamon

Appl. Math. Lett. Vol. 7, No. 3, pp. 57–62, 1994

0893-9659/94 $7.00 + 0.00

0893-9659(94)E0032-7

The Sharpness of a Pointwise Error Bound for the Fejér-Hermite Interpolation Process on Sets of Positive Measure

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(Received November 1993; accepted December 1993)

Abstract—Based on some important properties of the Chebyshev polynomials, an appropriate quantitative resonance principle is applied to establish the sharpness on sets of positive measure of a pointwise error bound, given by P. Vértesi (1971) in connection with the approximation by Fejér-Hermite polynomials on Lipschitz classes.

Keywords—Fejér-Hermite process, Pointwise error bounds, Quantitative resonance principle, Sharpness on sets of positive measure, Strongly mixing condition.

1. INTRODUCTION AND RESULTS

It is the purpose of this note to discuss, on sets of positive measure, the sharpness of a pointwise error estimate, given by Vértesi [1] for the approximation by Fejér-Hermite polynomials on Lipschitz classes (see also [2] as well as [3, p. 168ff] and the literature cited there). To this end, let $C[-1,1]$ be the Banach space of (real-valued) functions, continuous on the compact interval $[-1,1]$ of the real axis $\mathbb{R}$. For $f \in C[-1,1]$ and $n \in \mathbb{N}$ (set of natural numbers), the Fejér-Hermite polynomial $H_n(f,x) \in \mathcal{P}_{2n-1}$ (set of algebraic polynomials of degree less than or equal to $2n-1$) is uniquely determined by the conditions $(1 \le j \le n)$

$$H_n\left(f, \xi_j^{(n)}\right) = f\left(\xi_j^{(n)}\right), \qquad H_n'\left(\xi_j^{(n)}\right) = 0,$$

the Chebyshev knots $\left(\xi_j^{(n)}\right)$ being given by

$$\xi_j^{(n)} := \cos\frac{2j-1}{2n}\pi \qquad \text{for } 1 \le j \le n, \quad n \in \mathbb{N}.$$

In [1], the following direct estimate was shown: for each $f \in \mathrm{Lip}\gamma$, where $(0 < \gamma \le 1)$

$$\begin{aligned} \mathrm{Lip}\,\gamma &:= \{f \in C[-1,1] : \omega(f,t) = O(t^\gamma), t \to 0+\}, \\ \omega(f,t) &:= \sup\{|f(x+h) - f(x)| : x, x+h \in [-1,1], 0 \le h \le t\}, \end{aligned}$$

there exists a constant $M < \infty$ such that $(x \in [-1,1], n \in \mathbb{N})$

$$|H_n(f,x) - f(x)| \le M\left[\left(\frac{\sqrt{1-x^2}}{n}\right)^\gamma + \frac{1}{n}\right]. \tag{1.1}$$

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This pointwise error bound is sharp in the following sense.

THEOREM 1. *For each $0 < \gamma < 1$ and $0 < \varepsilon (< 1)$, there exists a counterexample $f_{\gamma,\varepsilon} \in Lip\gamma$ such that for almost every point $x \in [-1+\varepsilon, 1-\varepsilon]$*

$$\limsup_{n\to\infty} n^{\gamma} |H_n(f_{\gamma,\varepsilon}, x) - f_{\gamma,\varepsilon}(x)| \geq \frac{1}{2}\left(1-x^2\right)^{\gamma/2}. \tag{1.2}$$

Note that the term $1/n$ in the upper bound (1.1) is still unreflected in the lower bound (1.2). Moreover, it would be desirable to have (1.2) almost everywhere on the whole interval $[-1,1]$. In this regard, it may be worthwhile to mention the following partial result.

THEOREM 2. *For each $0 < \gamma < 1$, there exists $f_\gamma \in C[-1,1]$ for which (1.2) holds true for almost all $x \in [-1,1]$.*

Proofs of these assertions will be given in the next section. The arguments heavily rely on some important properties of the Chebyshev polynomials $T_n(x) := \cos(n \arccos x)$ (of the first kind) as well as on an appropriate quantitative extension of the classical resonance principle, dealing, however, with error functionals which depend on arbitrary index sets.

2. PROOFS

Let us briefly recall the resonance principle mentioned. For a (real) Banach space X with norm $\|\cdot\|_X$, let X^* denote the set of (real-valued) sublinear, bounded functionals V on X, thus,

$$|V(f+g)| \leq |Vf| + |Vg|, \qquad |V(af)| = |a|\,|Vf|$$

for $f, g \in X, a \in \mathbb{R}$ and

$$\|V\|_{X^*} := \sup\left\{|Vf| : \|f\|_X \leq 1\right\} < \infty.$$

Let ω be an abstract modulus (of continuity), i.e., $\omega \in C[0,\infty)$ with

$$0 = \omega(0) < \omega(t) \leq \omega(s+t) \leq \omega(s) + \omega(t) \qquad (0 < s, t),$$

which we additionally assume to satisfy $\lim_{t\to 0+} t^{-1}\omega(t) = \infty$ (e.g., $\omega(t) = t^\gamma$ for $0 < \gamma < 1$). In these terms, one has the following principle.

RESONANCE PRINCIPLE.

For arbitrary index sets $\mathbb{A}, \mathbb{B}_n, n \in \mathbb{N}$, and $\mathbb{T}_\alpha, \alpha \in \mathbb{A}$, consider a measure of smoothness $\{U_{t,\alpha} \in X^* : t \in \mathbb{T}_\alpha, \alpha \in \mathbb{A}\}$ and a family of error functionals $\{V_{n,\beta} \in X^* : \beta \in \mathbb{B}_n, n \in \mathbb{N}\}$ together with some function $\sigma(t) > 0$ for $t \in \mathbb{T}_\alpha, \alpha \in \mathbb{A}$, and a strictly decreasing nullsequence $(\tau_n)_{n=1}^\infty \subset \mathbb{R}$. Suppose there exists a sequence $(g_n)_{n=1}^\infty \subset X$ of test elements such that ($n \in \mathbb{N}$)

$$\|g_n\|_X \leq C_1, \tag{2.1}$$

$$|U_{t,\alpha} g_n| \leq C_2 \min\left\{1, \frac{\sigma(t)}{\tau_n}\right\}, \qquad (t \in \mathbb{T}_\alpha, \alpha \in \mathbb{A}), \tag{2.2}$$

$$\|V_{n,\beta}\|_{X^*} \leq C_{3,n}, \qquad (\beta \in \mathbb{B}_n), \tag{2.3}$$

$$|V_{n,\beta} g_j| \leq C_{4,\beta} C_{5,j} \tau_n, \qquad (1 \leq j \leq n-1,\ \beta \in \mathbb{B}_n), \tag{2.4}$$

$$|V_{n,\beta} g_n| \geq C_{6,\beta} > 0, \qquad (\beta \in \mathbb{B}_n). \tag{2.5}$$

Then, to each abstract modulus ω, there exists a (strictly increasing) subsequence $(n_k)_{k=1}^\infty \subset \mathbb{N}$ and a counterexample $f_\omega \in X$ such that

$$|U_{t,\alpha} f_\omega| \leq 6C_2\,\omega(\sigma(t)) \qquad (t \in \mathbb{T}_\alpha, \alpha \in \mathbb{A}), \tag{2.6}$$

$$\limsup_{n\to\infty} \frac{|V_{n,\beta} f_\omega|}{\omega(\tau_n)} \geq C_{6,\beta} \quad \text{for all } \beta \in \limsup_{k\to\infty} \mathbb{B}_{n_k} := \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \mathbb{B}_{n_j}. \tag{2.7}$$

For a proof via a gliding hump method, see [4] (see also [5, 3, p. 134ff], particularly in connection with the condensation of singularities on a limes superior of (arbitrary) index sets). Indeed, it follows that conditions (2.1), (2.2) imply (2.6), whereas (2.7) is a consequence of (2.1) and (2.3)–(2.5).

In connection with Theorem 1 and 2, in order to ensure that a limes superior of sets is in fact a set of full measure, we rely on some important properties of the Chebyshev polynomials T_n. To this end, let the measure μ be given by

$$\mu(A) := \int_A \frac{d\lambda(x)}{\sqrt{1-x^2}}, \qquad \text{for } A \in [-1,1] \cap \mathcal{B},$$

where λ denotes the Lebesgue measure and $[-1,1] \cap \mathcal{B}$ is the collection of all Borel subsets of $[-1,1]$. Then, each polynomial $T_n, n \geq 1$ is μ-measure preserving in the sense that (cf. 6, p. 205])

$$\mu\left(T_n^{-1}(A)\right) = \mu(A), \qquad \text{for each} \quad A \in [-1,1] \cap \mathcal{B}.$$

Moreover, the sequence (T_n) is strongly mixing, thus (cf. [6, p. 205])

$$\lim_{n\to\infty} \mu\left(T_n^{-1}(A) \cap B\right) = \frac{1}{\pi}\mu(A)\mu(B), \qquad \text{for all } A, B \in [-1,1] \cap \mathcal{B}. \tag{2.8}$$

From these properties, it easily follows (cf. [4]) that for every subsequence $(n_k) \subset \mathbb{N}$ and $A \in [-1,1] \cap \mathcal{B}$ with $\lambda(A) > 0$, one has

$$\lambda([-1,1] \setminus \limsup_{k\to\infty} T_{n_k}^{-1}(A)) = 0, \tag{2.9}$$

i.e., $\limsup_{k\to\infty} T_{n_k}^{-1}(A)$ is a set of full measure (in $[-1,1]$). Indeed, since for each $B \in [-1,1] \cap \mathcal{B}$, the condition $\lambda(B) > 0$ necessarily implies $\mu(B) > 0$, it is sufficient to show that

$$\mu([-1,1] \setminus \limsup_{k\to\infty} T_{n_k}^{-1}(A)) = 0,$$

which in turn, because of

$$[-1,1] \setminus \limsup_{k\to\infty} T_{n_k}^{-1}(A) = \bigcup_{k=1}^{\infty} \left([-1,1] \setminus \bigcup_{j=k}^{\infty} T_{n_j}^{-1}(A)\right),$$

would be a consequence of

$$\mu\left([-1,1] \setminus \bigcup_{j=k}^{\infty} T_{n_j}^{-1}(A)\right) = 0, \qquad \text{for each } k \in \mathbb{N}.$$

But the latter assertion in fact holds true, since $\mu(A) > 0$ and since for $i \geq k$

$$T_{n_i}^{-1}(A) \cap \left([-1,1] \setminus \bigcup_{j=k}^{\infty} T_{n_j}^{-1}(A)\right) = \emptyset,$$

thus, in view of (2.8) for every $k \in \mathbb{N}$

$$0 = \lim_{i\to\infty} \mu\left(T_{n_i}^{-1}(A) \cap \left([-1,1] \setminus \bigcup_{j=k}^{\infty} T_{n_j}^{-1}(A)\right)\right) = \frac{1}{\pi}\mu(A)\mu\left([-1,1] \setminus \bigcup_{j=k}^{\infty} T_{n_j}^{-1}(A)\right).$$

PROOF OF THEOREM 1. To proceed via the resonance principle, given $0 < \gamma, \varepsilon < 1$, consider

$$\begin{aligned}
X &= C[-1,1], && \text{with } \|f\|_C := \max\{|f(u)| : u \in [-1,1]\},\\
\mathbb{A} &= [-1,1), \mathbb{T}_\alpha = (0, 1-\alpha], && \text{for } \alpha \in \mathbb{A},\\
\mathbb{B}_n &= [-1+\varepsilon, 1-\varepsilon] \cap T_n^{-1}\left(\left[\frac{1}{2},1\right]\right), && \text{for } n \in \mathbb{N},
\end{aligned}$$

$$\sigma(t) = t, \qquad \tau_n = \frac{1}{n}, \qquad \omega(t) = t^\gamma,$$

$$U_{t,\alpha} f = |f(\alpha+t) - f(\alpha)|, \qquad V_{n,\beta} f = |H_n(f,\beta) - f(\beta)|,$$

$$g_n(u) = T_n(u)\left(1-u^2\right)^{\gamma/2} h(u),$$

where $h(u) = h_\varepsilon(u)$ denotes a function, arbitrarily often differentiable on $[-1,1]$ with the properties $0 \le h(u) \le 1$ for $u \in [-1,1]$ and

$$h(u) = \begin{cases} 1, & u \in [-1+\varepsilon, 1-\varepsilon] \\ 0, & u \in \left[-1, \frac{-1+\varepsilon}{2}\right] \cup \left[\frac{1-\varepsilon}{2}, 1\right]. \end{cases}$$

Obviously, $g_n \in C[-1,1]$ with $\|g_n\|_C \le 1$, thus (2.1) with $C_1 = 1$. Moreover,

$$\begin{aligned}
|g_n'(u)| &\le |T_n'(u)\left(1-u^2\right)^{\gamma/2} h(u)| + |T_n(u)\gamma u\left(1-u^2\right)^{-1+\gamma/2} h(u)| + |T_n(u)\left(1-u^2\right)^{\gamma/2} h'(u)| \\
&=: A_1(u) + A_2(u) + A_3(u),
\end{aligned}$$

say. Since h vanishes outside $I_\varepsilon := [-1+\varepsilon/2, 1-\varepsilon/2]$, the test elements g_n are in fact arbitrarily often differentiable on $[-1,1]$, and one has that for $u \in [-1,1]$

$$\begin{aligned}
A_1(u) &\le \sup_{y \in I_\varepsilon} |T_n'(y)| \le \sup_{y \in I_\varepsilon} \frac{n}{\sqrt{1-y^2}} =: M_{1,\varepsilon} \cdot n, \\
A_2(u) &\le \sup_{y \in I_\varepsilon} \left(1-y^2\right)^{-1+\gamma/2} =: M_{2,\varepsilon}, \qquad A_3(u) \le \|h'\|_C.
\end{aligned}$$

Therefore, with some constant $M = M_\varepsilon := M_{1,\varepsilon} + M_{2,\varepsilon} + \|h'\|_C$,

$$\|g_n'\|_C \le Mn, \qquad \text{for all } n \in \mathbb{N}. \tag{2.10}$$

Obviously, $U_{t,\alpha} \in (C[-1,1])^*$ is well-defined for each $t \in \mathbb{T}_\alpha$, $\alpha \in \mathbb{A}$, and in view of (2.10)

$$U_{t,\alpha} g_n \le \begin{cases} 2\|g_n\|_C \le 2 \\ t\|g_n'\|_C \le Mtn \end{cases} \qquad \text{for } t \in \mathbb{T}_\alpha, \quad \alpha \in \mathbb{A}, \quad n \in \mathbb{N},$$

thus, (2.2) with $C_2 = \max\{2, M\}$. Furthermore, $V_{n,\beta} \in (C[-1,1])^*$ for every $\beta \in [-1,1]$, $n \in \mathbb{N}$ as well, and since the Fejér-Hermite process is positive and linear, (2.3) follows with

$$\|V_{n,\beta}\|_{C^*} \le 2 = C_{3,n}, \qquad \text{for all } \beta \in [-1,1],\ n \in \mathbb{N}.$$

Concerning condition (2.4), a Taylor expansion of g_i at β delivers

$$V_{n,\beta} g_j \le \|g_j'\|_C \left|H_n\left((u-\beta), \beta\right)\right| + \frac{1}{2}\|g_j''\|_C \left|H_n\left((u-\beta)^2, \beta\right)\right|.$$

Because of the well-known representation

$$H_n(f,x) = \frac{1}{n^2}\sum_{j=1}^{n} f\left(\xi_j^{(n)}\right)\left(\frac{T_n(x)}{x-\xi_j^{(n)}}\right)^2\left(1-\xi_j^{(n)}x\right), \tag{2.11}$$

it immediately follows that for $\beta \in [-1,1]$

$$H_n\left((u-\beta)^2,\beta\right) = \frac{1}{n^2}T_n(\beta)^2\sum_{j=1}^{n}\left(1-\xi_j^{(n)}\beta\right) \le \frac{2}{n},$$

$$|H_n\left((u-\beta),\beta\right)| = \left|\frac{1}{n^2}T_n(\beta)\left(1-\beta^2\right)T_n'(\beta)+\frac{1}{n}\beta T_n(\beta)^2\right| \le \frac{2}{n}.$$

Therefore, (2.4) holds true with $C_{4,\beta} = 1$, $C_{5,j} = 2\left\|g_j'\right\|_C + \left\|g_j''\right\|_C$. Finally, since $H_n g_n = 0$ because of $g_n\left(\xi_j^{(n)}\right) = 0$ for $1 \le j \le n$ (cf. (2.11)) and since $|T_n(\beta)| \ge 1/2$, $h(\beta) = 1$ for $\beta \in \mathbb{B}_n$ by definition, one has

$$V_{n,\beta}g_n = |g_n(\beta)| = \left|T_n(\beta)\left(1-\beta^2\right)^{\gamma/2}h(\beta)\right| \ge \frac{1}{2}\left(1-\beta^2\right)^{\gamma/2},$$

thus, (2.5) with $C_{6,\beta} = 2^{-1}\left(1-\beta^2\right)^{\gamma/2} > 0$. Hence, the resonance principle may be applied to establish the existence of a counterexample $f_{\gamma,\varepsilon} \in C[-1,1]$ and of a subsequence $(n_k) \subset \mathbb{N}$ such that

$$|f_{\gamma,\varepsilon}(\alpha+t) - f_{\gamma,\varepsilon}(\alpha)| \le 6\max\{2,M\}t^\gamma$$

for all $\alpha \in [-1,1)$, $t \in (0, 1-\alpha]$, thus $f_{\gamma,\varepsilon} \in \mathrm{Lip}\,\gamma$, and

$$\limsup_{n\to\infty} n^\gamma\,|H_n(f_{\gamma,\varepsilon},\beta) - f_{\gamma,\varepsilon}(\beta)| \ge \frac{1}{2}\left(1-\beta^2\right)^{\gamma/2}$$

for all $\beta \in \limsup_{k\to\infty}\mathbb{B}_{n_k}$. Since by (2.9) for $A = [1/2,1]$

$$\lambda\left([-1+\varepsilon,1-\varepsilon]\setminus\limsup_{k\to\infty}\mathbb{B}_{n_k}\right) = \lambda\left([-1+\varepsilon,1-\varepsilon]\setminus\limsup_{k\to\infty}T_{n_k}^{-1}\left(\left[\frac{1}{2,1}\right]\right)\right)$$
$$\le \lambda\left([-1,1]\setminus\limsup_{k\to\infty}T_{n_k}^{-1}\left(\left[\frac{1}{2,1}\right]\right)\right) = 0,$$

this establishes Theorem 1 completely ∎

PROOF OF THEOREM 2.

Essentially we proceed as for Theorem 1, but through a nullsequence ε_n, $n \in \mathbb{N}$. Then, with

$$\widetilde{\mathbb{B}}_n = [-1+\varepsilon_n, 1-\varepsilon_n] \cap T_n^{-1}\left(\left[\frac{1}{2,1}\right]\right), \qquad \tilde{g}_n(u) = T_n(u)\left(1-u^2\right)^{\gamma/2}h_{\varepsilon_n}(u),$$

one may establish (2.1), (2.3), (2.4) as well as (2.5) for $\beta \in \widetilde{\mathbb{B}}_n$, whereas the proof of (2.2) fails since $M_{\varepsilon_n} \ne O(1)$ (essentially due to $M_{1,\varepsilon}$). Nevertheless, the resonance principle may be applied to show the existence of an element $f_\gamma \in C[-1,1]$ and a subsequence $(n_k) \subset \mathbb{N}$ satisfying (2.7), thus,

$$\limsup_{n\to\infty} n^\alpha\,|H_n(f_\gamma,\beta) - f_\gamma(\beta)| \ge \frac{1}{2}\left(1-\beta^2\right)^{\gamma/2}$$

for all $\beta \in \limsup_{k\to\infty}\widetilde{\mathbb{B}}_{n_k}$. Let $\beta \in (-1,1) \cap \limsup_{k\to\infty} T_{n_k}^{-1}([1/2,1])$ be arbitrary. Then there exists $j_0 = j_0(\beta) \in \mathbb{N}$ such that $\beta \in \left[-1+\varepsilon_{n_j}, 1-\varepsilon_{n_j}\right]$ for every $j \ge j_0$, and therefore,

$$\beta \in \bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty}\left(\left[-1+\varepsilon_{n_{j_0}}, 1-\varepsilon_{n_{j_0}}\right] \cap T_{n_j}^{-1}\left(\left[\frac{1}{2,1}\right]\right)\right)$$
$$\subset \bigcap_{k=j_0}^{\infty}\bigcup_{j=k}^{\infty}\left(\left[-1+\varepsilon_{n_j}, 1-\varepsilon_{n_j}\right] \cap T_{n_j}^{-1}\left(\left[\frac{1}{2,1}\right]\right)\right) = \limsup_{k\to\infty}\widetilde{\mathbb{B}}_{n_k}.$$

In view of (2.9), this implies

$$\lambda\left((-1,1)\setminus \limsup_{k\to\infty}\widetilde{\mathbb{B}}_{n_k}\right)\leq \lambda\left((-1,1)\setminus \limsup_{k\to\infty}T_{n_k}^{-1}\left(\left[\frac{1}{2},1\right]\right)\right)=0,$$

which completes the proof of Theorem 2. ∎

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